☆ equivariant vs Kähler parameters

<u>Higgs</u> ∘ equivariant

$$\mathcal{M}_H = M /\!/\!/ G_c = \mu_{\mathbb{C}}^{-1}(0) /\!/ G$$

Suppose $\tilde{G} \curvearrowright M \quad \Rightarrow \quad G_F := \tilde{G}/G \curvearrowright \mathcal{M}_H$, $\quad G \triangleleft \tilde{G}$

G_F: flavor symmetry group

an element of $\mathrm{Lie}\, G_F$ is called mass parameter
equiv.

<u>ex</u> quiver gauge theory $\quad \tilde{G} = G \times \prod_{i \in Q_0} GL(W_i)$

∘ Kähler/deformation parameter

Suppose $G = \prod_{i \in I} GL(n_i)$ for brevity

$\mu_{\mathbb{C}} : M \longrightarrow \mathrm{Lie}\, G^*$

$\ni \zeta_{\mathbb{C}}$ s.t. $\mathrm{Ad}_g^*(\zeta_{\mathbb{C}}) = \zeta_{\mathbb{C}}$

$\zeta_{\mathbb{C}} = \bigoplus \zeta_i \, \mathrm{id}_{n_i} \quad \zeta_i \in \mathbb{R}^3$

$$\mu^{-1}(\zeta_{\mathbb{C}}) /\!/_{\zeta_{\mathbb{R}}} G$$

$\zeta_{\mathbb{C}}$: cpx param. $\quad \zeta_{\mathbb{R}}$: Kähler param.

<u>Coulomb</u>

∘ equiv. $\quad \pi_1(G) = \pi_1(\prod GL(n_i)) = \mathbb{Z}^I$

$\pi_1(G) = \pi_0(\mathrm{Gr}_G) = \pi_0(\mathcal{R})$

$\therefore H_*^{G_{\mathcal{O}}}(\mathcal{R})$ is $\pi_1(G) \simeq \mathbb{Z}^I$-graded

$$\therefore \mathcal{M}_C \curvearrowleft T^I = \mathrm{Hom}(\pi_1(G), \mathbb{C}^\times)$$

∘ deformation / resolution

$$G \triangleleft \tilde{G} \curvearrowright N \qquad \tilde{G}_{\mathcal{O}} \curvearrowright \mathcal{R}$$

$\Rightarrow$ $H_*^{\tilde{G}_{\mathcal{O}}}(\mathcal{R})$ is a deformation of $H_*^{G_{\mathcal{O}}}(\mathcal{R})$

parametrised by $H^*_{G_F}(pt) = \mathbb{C}[\mathrm{Lie}\, G_F]$

suppose $G_F = \tilde{G}/G$ is torus

$$\Rightarrow \quad H_*^{\tilde{G}}(\mathcal{R}_{\tilde{G},N}) \curvearrowleft \pi_1(\tilde{G})^\wedge \supset \pi_1(G_F)^\wedge = G_F^\vee$$

dual torus

$$\text{Then} \quad H_*^{\tilde{G}}(\mathcal{R}_{\tilde{G},N})^{G_F^\vee} = H_*^{G_{\mathcal{O}}}(\mathcal{R})$$

i.e. $\mathcal{M}_C(\tilde{G},N)/\!/G_F^\vee$ is a deformation

$$\begin{array}{c} \mathcal{M}_C(\tilde{G},N)/\!/G_F^\vee \\ \downarrow \\ \mathrm{Lie}\, G_F = (\mathrm{Lie}\, G_F^\vee)^* \end{array} \longleftarrow \text{moment map}$$

(partial) resolution can be constructed by considering GIT quotient

motivation/conjecture

"Conjecture" We have 3d TQFT for the gauge theory (G, M)

X^3 : 3-mfd without ∂ $\rightsquigarrow$ $Z_{G,M}(X)$ "$\in$" $\mathbb{C}$ need to be corrected
oriented

Σ : 2-mfd without ∂ $\rightsquigarrow$ $Z_{G,M}(\Sigma)$: Hilbert space
oriented

$\partial X^3 = \Sigma \quad \rightsquigarrow \quad Z_{G,M}(X) \in Z_{G,M}(\Sigma)$

+ gluing axiom , $Z(-\Sigma) = Z(\Sigma)$ etc

Phys. def $\Rightarrow$ Gauge$(G, M) \cong$ σ-model to $\mathcal{M}_C$,

$\therefore$ $Z_{G,M} = Z_{\sigma\text{-model}} =$ Rozansky-Witten theory

<u>Fact</u> $Z_{\sigma\text{-model}}(S^2) = \bigoplus_{q} H^q(\mathcal{M}, \mathcal{O}_{\mathcal{M}})$
with target $\mathcal{M}$

"<u>Cor</u>"1 $Z_{G,M}(S^2) = \mathbb{C}[\mathcal{M}_C]$ as $\mathcal{M}_C$: affine variety

<u>Rem</u>. RW : it is "hazardous" to apply this definition to non compact $\mathcal{M}$

"<u>Cor</u>" 2 $Z_{G,M}(S^2 \times S^1) = \dim Z_{G,M}(S^2) = \dim \mathbb{C}[\mathcal{M}_C] = \infty$

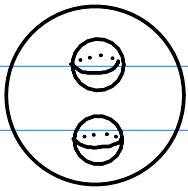
Recall $S^1 \curvearrowright \mathcal{M}_C$ $\therefore$ $\mathbb{C}[\mathcal{M}_C]$: S^1-module

So $\operatorname{ch} \mathbb{C}[\mathcal{M}_C] = \sum_m t^m \dim \mathbb{C}[\mathcal{M}_C]_m$ weight m
could be well-defined

e.g. $\mathcal{M}_C = \mathbb{C}^2 = \operatorname{Spec} \mathbb{C}[x, y]$ $\quad x, y$: wt $= 1$

$\operatorname{ch} \mathbb{C}[\mathcal{M}_C] = 1 + 2t + 3t^2 + \cdots\cdots$

<u>Fact</u> Casson$(S^2 \times S^1) = -\frac{1}{12} = \zeta(-1)$ So match with

$X^3 =$  $\Rightarrow Z_{M,G}(X) \in \mathrm{Hom}(Z_{M,G}(S^2)^{\otimes 2}, Z_{M,G}(S^2))$

commutative multiplication

$\therefore \mathcal{M}_C = \mathrm{Spec}\,(Z_{M,G}(S^2),\ \mathrm{mult} = Z_{M,G}(X))$

So enough to define

In mathematical approach to $Z_{G,M}$, people use moduli spaces of SW type nonlinear PDE on X or Σ

Riemann surface Σ

A : G_c-connection on P : G_c-bundle

Φ : section of $K_\Sigma^{1/2} \otimes (P \times_{G_c} M)$

equation

- $\bar\partial_A \Phi = 0$
- $\mu_{\mathbb{C}}(\Phi) = 0$
- $\mu_{\mathbb{R}}(\Phi) = * F_A$

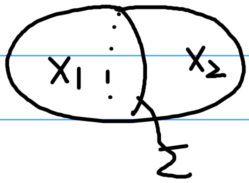
modulo gauge equiv.

$\mathcal{M}_X$ = moduli space of sol. on X $\quad Z_{G,M}(X) = \#^{\mathrm{vir}} \mathcal{M}_X$

$\mathcal{M}_\Sigma =$ 〃 Σ $\quad Z_{G,M}(\Sigma) = H^*(\mathcal{M}_\Sigma)$

$\partial X = \Sigma \rightarrow \mathcal{M}_X \xrightarrow{\text{bdry}} \mathcal{M}_\Sigma$ $\quad [\mathcal{M}_X] \in H^*(\mathcal{M}_\Sigma)$

"lagrangian"

$X =$  $\quad \#^{\mathrm{vir}} \mathcal{M}_X = \langle [\mathcal{M}_{X_1}], [\mathcal{M}_{X_2}] \rangle$

This construction was worked out (partially)

if $(G = SL(2), M = 0)$, $(G = U(1), M = \mathbb{C} \oplus \mathbb{C}^*)$

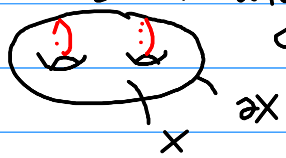
instantanton Floer $\qquad$ Heegard Floer

$\mathcal{M}_\Sigma$ = moduli of flat connections $= \mathrm{Hom}(\pi_1(\Sigma), SU(2))/\mathrm{conj}$

$\mathcal{M}_X = \mathrm{Hom}(\pi_1(X), SU(2))/\mathrm{conj}.$

$\mathcal{M}_\Sigma$ = moduli of line b'dle + section $= S^g \Sigma_g$

$\mathcal{M}_X$ = image of attaching cycles



But $\Sigma = S^2$ $\quad \mathcal{M} \overset{SU(2)}{\underset{U(1)}{=}} pt / SU(2)$ $\quad H^*_{SU(2)}(pt) = \mathbb{C}/\mathbb{Z}_2$

$\quad = pt$ $\qquad$ not correct!

So need a correction!